Instability of a viscous liquid of variable density in a vertical Hele-Shaw cell

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Approximate equations of motion, continuity and mass transport are given for a viscous liquid of variable density moving very slowly between vertical and impermeable parallel planes. These equations are used to calculate approximate stability criteria when the liquid is at rest under a vertical density gradient. The results are applicable to the problem of the stability of a viscous liquid of variable density to two-dimensional disturbances in a porous medium.

An exact stability analysis for the liquid between parallel planes is also given, and expansions in powers of the disturbance wave-number are obtained for the critical Rayleigh number at neutral stability. The previous approximate results are found to correspond to the leading terms of the series expansions. For the most unstable type of disturbance, the velocity distribution closely resembles plane Poiseuille flow, which was the form assumed in the approximate equations.

An asymptotic expansion is derived for the critical Rayleigh number at neutral stability in a long vertical channel, or duct, the cross-section of which is a thin rectangle. The typical neutral disturbance possesses a 'boundary layer' at each end of the cell cross-section, and this has a small stabilizing effect.

The critical Rayleigh number for a long vertical channel of rectangular cross-section is found experimentally by comparing the density gradient of the liquid in the channel at neutral stability with the corresponding density gradient in a vertical capillary tube. There is better agreement with the exact theory than with the approximate theory, the experimental result being about 4 % higher than the value predicted by the 'exact' asymptotic expression, and about 10 % higher than the value predicted by the simple approximate theory.

1. Introduction

It is well known (see Lamb 1932, §330) that the flow of a viscous liquid at very low Reynolds numbers between parallel planes may be described by approximate equations of motion which have a simple form. Consider, in particular, vertical parallel planes which are separated by a small distance $2h$. Take the origin of a rectangular co-ordinate system (X, Y, Z) midway between the planes, with OY normal to them and with OZ directed vertically upwards. It will be assumed, first, that the component of velocity normal to the planes vanishes, so that the pressure P is a function of X and Z alone, and secondly, that the components of velocity in the directions of OX and OZ are slowly varying functions of those co-ordinates. Then the velocity distribution is nearly parabolic in the

Y-co-ordinate, and if mean values are taken by integrating with respect to Y, the mean-velocity vector is everywhere parallel to the hydraulic head. The approximate equations of motion are

$$u = -\frac{h^2}{3\mu}\frac{\partial P}{\partial X}, \quad w = -\frac{h^2}{3\mu}\left(\frac{\partial P}{\partial Z}+\rho g\right), \tag{1}$$

where u and w are the components of the mean velocity, P, ρ and μ denote respectively the mean pressure, density and viscosity of the liquid, and g is the acceleration due to gravity.

When ρ and μ are constants, equation (1) and the equation of continuity

$$\frac{\partial u}{\partial X}+\frac{\partial w}{\partial Z} = 0 \tag{2}$$

together show that the mean motion is irrotational and that a mean-velocity potential exists. Hele-Shaw (1898) has used this principle experimentally to render visible the streamlines of two-dimensional ideal potential flow about cylindrical bodies, and has treated, in a similar manner, lines of magnetic induction about elliptic cylinders and cylindrical shells (Hele-Shaw & Hay 1900).

Saffman & Taylor (1958) have observed that the equations (1) are identical with Darcy's law for two-dimensional flow of a viscous liquid through a porous medium of permeability $\frac{1}{3}h^2$. Using this analogy, they have made theoretical and experimental investigations of the stability of the interface between two immiscible viscous fluids in motion through a porous medium in a gravitational field.

Since the observation of flow phenomena within porous materials is difficult, the use of a Hele-Shaw analogue in certain experimental problems could offer attractive possibilities. The work described in this paper arose from the investigation of the instability of a liquid of variable density in equilibrium under gravity in a vertical tube filled with porous material (Wooding 1959). The Hele-Shaw analogue is a long vertical channel, with a horizontal cross-section in the form of a very narrow rectangle.

Exact calculations of the stability of a viscous liquid of variable density between parallel planes have been made by Ostrach (1955) and by Yih (1959). However, these authors did not discuss the most unstable forms of infinitesimal disturbance that can arise.

A more realistic approach has been adopted by Bentwich (1959), who examined the inhibiting effect of horizontal magnetic fields upon the gravitational instability of fluid in a vertical tube of square or rectangular cross-section. Bentwich did not extend his work to the case of either infinite parallel planes or a Hele-Shaw cell. As these examples are of some importance, the gravitational stability theory is discussed in this paper.

2. Approximate stability theory

When the space between two parallel planes is filled by a liquid of variable density, the nature of the density distribution is influenced by both the molecular diffusion and by mechanical dispersion arising from the slow motion of the fluid. The relative importance of these effects can be estimated by adapting the work of Taylor (1953, 1954*a*) and of Aris (1956) on mass transport of solute in tubes to the case of unidirectional flow between parallel planes. For simplicity, suppose that the fluid motion is in the Z-direction, and that the concentration of solute is a slowly varying function of Y and Z only. If gravity effects do not modify the velocity profile appreciably, it is found that the dissolved material is dispersed relative to a plane $Z = wt$, moving with the mean velocity w, by a process resembling molecular diffusion but with a diffusivity

$$\kappa + \frac{2}{105}\frac{w^2h^2}{\kappa},$$

where κ is the molecular diffusivity. This expression shows that, if the mean velocity is so small that the relation

$$\left(\frac{2hw}{\kappa}\right)^2 \ll 210$$

is satisfied, the effect of mechanical dispersion of the solute is negligible compared with molecular diffusion. When this condition applies, the density ρ can be treated as a slowly varying function of X and Z, and the approximate equations of motion and continuity, (1) and (2), will apply. A further equation, the equation of mass transport, is then found to be

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial X} + w\frac{\partial\rho}{\partial Z} = \kappa\left(\frac{\partial^2\rho}{\partial X^2} + \frac{\partial^2\rho}{\partial Z^2}\right), \tag{3}$$

where u and w are the X- and Z-components of the mean velocity, as before. The equations (1), (2) and (3) are identical with the corresponding equations of two-dimensional motion in a porous medium of permeability $\frac{1}{3}h^2$ and unit porosity, saturated by a liquid of effective diffusivity κ and of variable density ρ.

At this stage, the equations are seen to possess a shortcoming. Because inertial terms are neglected in (1), the system of equations is of first order in the time, and initial values of velocity and density cannot be assigned independently. Clearly, the neglected terms must be significant in the initial stages of the motion. However, it is easily shown that the inertial terms become negligible within a period of time of order h^2/ν seconds, where ν is the kinematic viscosity, after which time the equations (1) become valid.

Now consider a viscous liquid at rest between vertical parallel planes. For equilibrium, it is necessary that the unperturbed values of the pressure P and the density ρ should be functions of Z only. It will be assumed that the density gradient $d\rho/dZ$ is a constant, and that the total variation in density is sufficiently small for ρ, and quantities depending upon ρ such as viscosity and diffusivity, to be considered constant to first order.

It is convenient to introduce dimensionless spatial variables $(x, z) \equiv (X/h, Z/h)$ using h (equal to half the distance between the parallel planes) as the typical length unit.

If a small perturbation of arbitrary form is added to the equilibrium value of the density, the pressure field will be perturbed and a small velocity (u, w), a function of x, z and t, will appear. The linearized equations governing these small disturbances are determined from (1), (2) and (3). Suppose that one Fourier component of the density disturbance is of form $\cos \alpha x \cos \gamma z\, e^{\omega t}$, where α and γ are dimensionless wave-numbers in the x- and z-directions, respectively. Then the linearized perturbation equations form a compatible system provided that

$$\frac{h^2\omega}{\kappa} = -\alpha^2 - \gamma^2 + \frac{\alpha^2\lambda/3}{\alpha^2+\gamma^2}, \tag{4}$$

where

$$\lambda = \frac{dp}{dZ}\frac{gh^4}{\mu\kappa} \tag{5}$$

is the Rayleigh number.

Equation (4) is, of course, identical in form with the corresponding compatibility equation for the stability of a viscous liquid in a porous medium (Lapwood 1948; Wooding 1959). The exact correspondence follows when h is replaced by a length unit b, determined by boundary dimensions in the (X, Z)-plane. Dimensionless wave-numbers α_p, γ_p are defined with respect to b, giving $\alpha_p/b = \alpha/h$, and similarly for γ_p. In place of (5), one has

$$\lambda_p = \frac{d\rho}{dZ}\frac{g(h^2/3)\,b^2}{\mu\kappa} = \frac{\lambda}{3}\frac{b^2}{h^2},$$

where λ_p is defined as the Rayleigh number for a porous medium of permeability $\frac{1}{3}h^2$, saturated with a liquid, of viscosity μ, which diffuses through the porous material with an effective diffusivity κ.

At neutral equilibrium ($\omega = 0$), equation (4) gives

$$\lambda = \frac{3(\alpha^2+\gamma^2)^2}{\alpha^2}. \tag{6}$$

This formula shows that, when γ is held constant, the minimum value of λ arises for $\alpha = \gamma$, corresponding to square convection cells. However, if α is held constant, λ has a minimum value when $\gamma = 0$, the fluid motion being in the form of long vertical columns. The first case arises when the vertical Hele-Shaw cell is of finite height, but of infinite horizontal extent—a situation which is analogous to that of Rayleigh instability in a porous medium (Lapwood 1948). The second case arises when the vertical cell is of finite width but of infinite height, the situation being analogous to that of a long vertical tube filled with porous material (Wooding 1959). In each case, the boundary conditions along the edges of the strip must be expressed in an approximate form involving mean values of the density ρ and the velocity (u, w). At a horizontal edge, the vanishing of the horizontal velocity component u is ignored, since the effect of the non-slip condition becomes inappreciable at a short distance from the edge. Similarly, the vanishing of w is ignored at a vertical edge.

Some comparisons will be made later between the above approximate results and the results of 'exact' calculations in §§3 and 4.

3. Exact theory of neutral stability

The approximate stability theory, based upon equations (1), (2) and (3), is satisfactory provided that the wavelength of the disturbance in each of the X- and Z-directions is large compared with the spacing between the parallel planes.

At shorter wavelengths, the analysis breaks down because the disturbance is no longer of a quasi-two-dimensional character. The effects of shear stresses in the X- and Z-directions, and of molecular diffusion in the Y-direction, become apparent at short wavelengths, and one would expect to find the physical system rather more stable than the approximate theory predicts.

Yih (1959) has given a proof, along the same lines as Pellew & Southwell (1940) for Rayleigh instability, of the 'principle of the exchange of stabilities' for a viscous liquid confined by vertical walls of unlimited height and heated from below, when the disturbance is periodic in the vertical direction, and also for walls of finite height bounded at the ends by horizontal conducting planes. In the latter case, periodicity in the vertical direction is not a necessary assumption. Apart from the above restrictions, Yih's proof is quite general, and is applicable to the work which follows. Then, since the behaviour of all neutral or amplified disturbances is aperiodic with respect to time, one may impose the condition $\partial/\partial t = 0$ for neutral stability.

Taking Cartesian axes as before, with the origin O midway between the parallel planes, OZ directed vertically upwards and OY normal to the planes, one has the following differential equations governing a neutrally stable disturbance

$$\operatorname{div}\mathbf{u} = 0, \tag{7}$$

$$\mathbf{g}\vartheta - \operatorname{grad} p + \mu\nabla^2\mathbf{u} = 0, \tag{8}$$

$$w\frac{d\rho}{dZ} = \kappa\nabla^2\vartheta, \tag{9}$$

where ρ is the primary density distribution, ϑ and p are perturbations in the density and pressure, and $\mathbf{u} \equiv (u, v, w)$ is the velocity perturbation. In equation (8), $\mathbf{g} \equiv (0, 0, -g)$.

The method of solution used here will follow that of Hales (1937), who examined certain problems in the stability of liquid in vertical tubes.

To eliminate p, one takes the curl of (8), and obtains equations for the two non-zero components of the vorticity

$$\left.\begin{aligned} \mu\nabla^2\left(\frac{\partial w}{\partial Y} - \frac{\partial v}{\partial Z}\right) &= g\frac{\partial\vartheta}{\partial Y}, \\ \mu\nabla^2\left(\frac{\partial u}{\partial Z} - \frac{\partial w}{\partial X}\right) &= -g\frac{\partial\vartheta}{\partial X}. \end{aligned}\right\} \tag{10}$$

Elimination of u and v between (10) and (7) gives the usual result

$$\mu\nabla^4 w = g\nabla_1^2\vartheta, \tag{11}$$

where $\nabla_1^2 \equiv \nabla^2 - \partial^2/\partial Z^2$. Then, if the spatial differentiations are referred to non-dimensional variables $(x, y, z) \equiv (X/h, Y/h, Z/h)$, (9) and (11) can be combined to give the familiar equation

$$\nabla^6 w = \lambda\nabla_1^2 w, \tag{12}$$

and a similar equation for ϑ. Here λ is the Rayleigh number defined in (5).

At the vertical insulating walls, the boundary conditions give

$$u = v = w = 0 \quad \text{and} \quad \partial\vartheta/\partial y = 0 \quad \text{when} \quad y = \pm 1. \tag{13}$$

Separable solutions of the equations, of suitable form, can be found by assuming that a disturbance can be Fourier analysed into components with respect to both x and z. For example, let

$$\left.\begin{aligned} u &= U(y)\sin\alpha x\,\sin\gamma z,\\ v &= V(y)\cos\alpha x\,\sin\gamma z,\\ w &= W(y)\cos\alpha x\,\cos\gamma z,\\ \vartheta &= \theta(y)\cos\alpha x\,\cos\gamma z. \end{aligned}\right\} \tag{14}$$

These represent components of a disturbance of cellular form in the (x, z)-plane, having dimensionless wave-numbers α and γ, respectively, The relative phases of u, v, w and ϑ, in the x- and z-directions, are determined by the relations (10), etc.

From (12) and (14), $W(y)$ obeys the equation

$$(D^2-\alpha^2-\gamma^2)^3\,W = \lambda(D^2-\alpha^2)\,W, \tag{15}$$

where $D \equiv d/dy$. (When one puts $\alpha = 0$, this equation corresponds to the equation (40) used by Yih (1959). The disturbance is then of two-dimensional type, and depends upon y and z only.) Yih has shown that solutions of equation (15) are of form $\exp(\pm p_i y)$, where

$$p_i^2 = \alpha^2+\gamma^2+m_i \quad (i = 1, 2, 3), \tag{16}$$

m_i being a root of the indicial equation

$$m^3-\lambda m-\lambda\gamma^2 = 0. \tag{17}$$

The form of equation (15) is such as to entail separate treatment of even and odd solutions for $W(y)$. If it can be assumed that the disturbance of Hele-Shaw type introduced in §2 leads to the greatest instability, it appears that the even solution

$$W(y) = \sum_{i=1}^{3} A_i \cosh p_i y \tag{18}$$

is of greatest physical interest. For the corresponding odd solution, the hyperbolic cosines are replaced by hyperbolic sines.

In order to impose the boundary conditions (13), it is necessary to solve (9) for θ, and (10) for U and V, in terms of the solution (18) for W. Use is also made of the equation of continuity (7) and the symmetrical nature of the boundary conditions. It is found that

$$\left.\begin{aligned} \theta &= \frac{h^2}{\kappa}\frac{d\rho}{dZ}\sum_{i=1}^{3} A_i \frac{\cosh p_i y}{m_i},\\ \gamma U &= \alpha\lambda\left(C\cosh ay-\gamma^2\sum_{i=1}^{3} A_i\frac{\cosh p_i y}{m_i^3}\right),\\ -\gamma V &= \lambda\left(\frac{\alpha^2}{a^2}Ca\sinh ay-\gamma^2\sum_{i=1}^{3} A_i\frac{p_i\sinh p_i y}{m_i^3}\right), \end{aligned}\right\} \tag{19}$$

where $a = (\alpha^2+\gamma^2)^{\frac{1}{2}}$, and C is a constant. The boundary condition $U = 0$ on $y = \pm 1$ gives

$$C = \frac{\gamma^2}{\cosh\alpha}\sum_{i=1}^{3}\frac{A_i\cosh p_i}{m_i^3}.$$

If the boundary conditions $V = W = D\theta = 0$ on $y = \pm 1$ are now imposed, the resultant characteristic equation is of the form

$$\begin{vmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \\ N_1 & N_2 & N_3 \end{vmatrix} = 0, \tag{20}$$

where

$$\left.\begin{aligned} L_i &= p_i \tanh p_i - \alpha^2 \tanh a/a, \\ M_i &= p_i^2 - \alpha^2, \\ \text{and} \quad N_i &= m_i^2 p_i \tanh p_i. \end{aligned}\right\} \tag{21}$$

When the odd solution for $W(y)$ is considered, the same characteristic equation is obtained, but with $\cosh p_i$ and $\sinh p_i$ interchanged, and with $\tanh a$ replaced by $\coth a$. When α tends to zero (in this latter case) the characteristic equation reduces to that given by Yih (1959, equation (46)). While still discussing three-dimensional disturbances, Yih made the assumption that the horizontal component of velocity parallel to the boundary planes was identically zero. However, it is clear from (19) that U does not necessarily vanish.

The method of Yih can be used to show that, if α is assumed to have a fixed positive value, the critical Rayleigh number is stationary with respect to variations in γ when the wavelength of the disturbance in the z-direction tends to infinity. If the condition $\partial\lambda/\partial\gamma = 0$ is imposed, it is easily shown, from (16) and (17), that both $\partial p_i/\partial\gamma$ and $\partial m_i/\partial\gamma$ $(i = 1, 2, 3)$ contain γ as a factor. Then, from (21), each of $\partial L_i/\partial\gamma$, $\partial M_i/\partial\gamma$ and $\partial N_i/\partial\gamma$ contains γ as a factor. Following Yih, one differentiates the characteristic equation (20) with respect to γ and puts $\partial\lambda/\partial\gamma = 0$. Each of the three determinants so obtained contains a column proportional to

$$\left\{\frac{\partial L_i}{\partial\gamma}\ \frac{\partial M_i}{\partial\gamma}\ \frac{\partial N_i}{\partial\gamma}\right\} \quad (i = 1, 2, 3),$$

whence the sum of the three determinants must vanish for $\gamma = 0$. Thus the characteristic equation (20), and the condition $\partial\lambda/\partial\gamma = 0$, can be satisfied simultaneously for $\gamma = 0$. Since positive values of γ involve larger gradients of velocity and density than when $\gamma = 0$, it appears that the stationary point $\gamma = 0$ gives a minimum value of λ.

In the special case $\gamma = 0$, the roots of the cubic equation (17) reduce to

$$m_1 = \lambda^{\frac{1}{2}}, \quad m_2 = \lambda^{-\frac{1}{2}}, \quad m_3 = 0,$$

and the equations subsequent to (17) are found to simplify considerably. Equations (19) and (18), together with the boundary conditions, give

$$u = v = 0, \quad w = \left(\frac{\cosh py}{\cosh p} - \frac{\cosh qy}{\cosh q}\right) \cos \alpha x, \tag{22}$$

where $p = p_1 = (\alpha^2 + \lambda^{\frac{1}{2}})^{\frac{1}{2}}$, $q = p_2 = (\alpha^2 - \lambda^{\frac{1}{2}})^{\frac{1}{2}}$ and the equation (20) reduces to

$$p \tanh p + q \tanh q = 0. \tag{23}$$

For small α, λ may be expanded in a series of ascending powers of α^2. The leading term, say λ_0, of the series is given by solutions of the equation

$$\lambda_0^{\frac{1}{4}}(\tanh \lambda_0^{\frac{1}{4}} - \tan \lambda_0^{\frac{1}{4}}) = 0, \tag{24}$$

for which the root of smallest magnitude is zero. Then λ is of order α^2, and the transcendental functions in (23) may be expanded in series of powers of their arguments, leading to the result

$$\lambda = 3\alpha^2(1+\tfrac{8}{21}\alpha^2+O(\alpha^4)). \qquad (25)$$

A comparison with the result of §2—that $\lambda = 3\alpha^2$ when $\gamma = 0$—shows that the simple approximate analysis gives the first term of the series (25).

It is of interest to note that, when both α and γ are small but non-zero, an expansion of the characteristic equation (20) gives

$$\lambda = \frac{3(\alpha^2+\gamma^2)^2}{\alpha^2} + \text{terms of fourth order in } \alpha \text{ and } \gamma,$$

for the smallest root of λ, which corresponds to the equation (6) calculated from simple theory. If α has a constant value, this expression shows at once that the smallest root has a minimum value at $\gamma = 0$.

An expansion of w in (22) for the case $\lambda_0 = 0$ and $\gamma = 0$ leads to

$$w = \{-3^{\frac{1}{2}}\alpha(1-y^2)+O(\alpha^3)\}\cos\alpha x, \qquad (26)$$

that is, the velocity distribution is almost parabolic with respect to y. This supports the assumptions of §2 that the fluid motion resembles Hele-Shaw flow.

For odd solutions, in the case $\gamma = 0$, the hyperbolic cosines in (22) are replaced by hyperbolic sines, and the characteristic equation (23) is replaced by

$$\frac{\tanh p}{p}+\frac{\tanh q}{q} = 0, \qquad (27)$$

where p and q have the same meanings as before. If a series expansion in powers of α^2 is assumed to exist, the leading term (λ_0) of the expansion is found to be given by solutions of

$$\frac{\tanh\lambda_0^{\frac{1}{4}}+\tan\lambda_0^{\frac{1}{4}}}{\lambda_0^{\frac{1}{4}}} = 0. \qquad (28)$$

Here the smallest root cannot be equal to zero, and it is, in fact, approximately equal to $(0{\cdot}7528\pi)^4 \approx 31{\cdot}29$.

The numerical value $\lambda \approx 31{\cdot}29$ is given by Ostrach (1955) and by Yih (1959) as the critical value of the Rayleigh number for neutral stability. However, even when $\alpha = 0$, this mode of disturbance is not as unstable as that arising when one takes the zero root of equation (24) with α finite but small. A physical explanation for the difference is readily found. Ostrach and Yih consider disturbance modes of two-dimensional form, the motion being restricted to the (y, z)-plane. Factors contributing to the stability are shear stresses arising from velocity gradients in the y-direction and molecular diffusion in the y-direction. These factors apply for the most unstable disturbance of this type, when the fluid moves in two columns, one rising, say, in the range $0 < y < 1$, and the other descending in the range $0 > y > -1$. However, when the velocity disturbance is of Hele-Shaw type, the density perturbation is almost constant with respect to y; the physical factors contributing to the stability of the liquid are shear stresses induced by velocity gradients in the y-direction, and molecular diffusion in the x-direction. When the wave-number α tends to zero, the latter effect becomes negligible.

An alternative treatment of the case $\gamma = 0$ makes use of the fact that equation (12) simplifies if w is independent of z. The method was used originally by Sir Geoffrey Taylor for the problem of stability in a vertical tube. In the simplified form, (12) becomes

$$\nabla_1^2(\nabla_1^4 - \lambda)\, w = 0 \quad \left(\nabla_1^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

This expression states that $(\nabla_1^4 - \lambda)\, w$ is an harmonic function of x and y which vanishes at the boundaries (from (9) and (11) and the boundary condition $w = 0$) and therefore vanishes everywhere. Then the fourth-order equation

$$(\nabla_1^4 - \lambda)\, w = 0 \tag{29}$$

is equivalent to (12). This is the same differential equation as that governing the transverse displacement of a freely vibrating thin elastic plate.

When the boundaries are two vertical and parallel insulating planes, the boundary conditions (13) reduce to

$$w = \frac{\partial}{\partial y}(\nabla_1^2 w) = 0,$$

and solutions of equation (29) give the results described in equations (22) to (26).

It is perhaps worth noting that, if the boundaries are two vertical and parallel *conducting* planes, the even solutions of (29), with boundary conditions

$$w = \nabla_1^2 w = 0,$$

are of form

$$w = \cos ry \cos \alpha x,$$

where $r^2 + \alpha^2 = \lambda^{\frac{1}{2}}$ and $r = (n + \frac{1}{2})\pi$ $(n = 0, 1, 2, \ldots)$. The most unstable case arises when $n = 0$, and

$$\lambda = (\tfrac{1}{4}\pi^2 + \alpha^2)^2 \rightarrow \tfrac{1}{16}\pi^4 \approx 6{\cdot}09$$

when $\alpha \rightarrow 0$. In this example, λ tends to a finite limit because the effect of molecular diffusion to (and from) the conducting walls remains significant when α becomes very small. Again, the velocity distribution is approximately parabolic in the y-direction. Ostrach (1955) briefly examined the two-dimensional modes of disturbance between vertical parallel conducting walls in connexion with the thermal instability of a viscous fluid, but the above more unstable mode was not discussed.

4. Instability in a long vertical channel of rectangular section

Some consideration will now be given to the calculation of the criterion for neutral stability in a vertical Hele-Shaw cell which is closed by boundaries in the (x, z)-plane. Two basic boundary configurations have been mentioned briefly in §2.

Suppose that vertical parallel planes, of width $2b$ and spaced a distance $2h$ form the sides of a Hele-Shaw cell of great depth, the cell being closed by impermeable vertical boundaries at the sides $x = \pm L$ $(L = b/h \gg 1)$. The dimensionless wave-number γ in the z-direction will be assumed to be zero, but the wave-number α in the x-direction must be finite. As L is large, the problem of determining the Rayleigh number λ at neutral stability is similar to the problem of determining the characteristic frequency of a narrow rectangular plate, vibrating freely in the lowest transverse mode for which the average displacement is identically zero. From previous considerations, it would appear that a disturbance motion antisymmetric about $x = 0$, but symmetric about $y = 0$ (taking the origin of dimensionless co-ordinates midway between the planes, as before) will furnish the lowest eigenvalue for which the mean displacement of the plate, or the net vertical flux of the fluid, will vanish.

The appropriate differential equation for the vertical velocity w is given by (29), and the boundary conditions upon the four vertical insulating walls are

$$\left.\begin{aligned} w = \frac{\partial}{\partial x}(\nabla_1^2 w) = 0 \quad \text{on} \quad x = \pm L,\\ w = \frac{\partial}{\partial y}(\nabla_1^2 w) = 0 \quad \text{on} \quad y = \pm 1.\end{aligned}\right\} \tag{30}$$

This problem can be treated by Taylor's classical method of expansion in Fourier series (Taylor 1933). Consider the expression

$$w = \sum_{n=0}^{\infty}\left\{A_n\left(\frac{\cosh p_n y}{\cosh p_n} - \frac{\cosh q_n y}{\cosh q_n}\right)\sin\frac{a_n x}{L} + B_n\left(\frac{\sinh r_n x}{r_n \cosh r_n L} + \frac{\sinh s_n x}{s_n \cosh s_n L}\right)\cos a_n y\right\}, \tag{31}$$

where $a_n = (n+\frac{1}{2})\pi$, and A_n and B_n are constants. Each term in (31) satisfies the boundary conditions $(\partial/\partial x)\nabla_1^2 w = 0$ on $x = \pm L$ and $w = 0$ on $y = \pm 1$, and the differential equation (29) is satisfied provided that

$$\left.\begin{aligned} p_n^2 &= a_n^2/L^2 + \lambda^{\frac{1}{2}}, & q_n^2 &= a_n^2/L^2 - \lambda^{\frac{1}{2}},\\ r_n^2 &= a_n^2 + \lambda^{\frac{1}{2}}, & s_n^2 &= a_n^2 - \lambda^{\frac{1}{2}}.\end{aligned}\right\} \tag{32}$$

It is necessary to determine non-trivial values of the constants A_n and B_n in such a way that (31) satisfies the conditions $w = 0$ on $x = \pm L$ and $(\partial/\partial y)\nabla_1^2 w = 0$ on $y = \pm 1$. This will be possible only if a certain characteristic equation, which relates λ and L, can be satisfied. The values of A_n and B_n are determined as follows. Since w is an even function of y which vanishes on the boundaries $y = \pm 1$, it is clear that w may be represented on the boundaries $x = \pm L$ by a Fourier series in $\cos a_m y$, where $m = 0, 1, 2, \ldots$. The vanishing of w on $x = \pm L$ then requires that each coefficient of the Fourier series so obtained should be zero. Similarly, $(\partial/\partial y)\nabla_1^2 w$ may be represented on the boundaries $y = \pm 1$ by a Fourier series in $\sin a_m x/L$. The vanishing of $(\partial/\partial y)\nabla_1^2 w$ on $y = \pm 1$ then requires that each Fourier coefficient be zero. Proceeding in this way, one obtains two infinite systems of equations

$$\left.\begin{aligned} \sum_{n=0}^{\infty}(K_{mn}A_n + L_{mn}B_n) = 0,\\ \sum_{n=0}^{\infty}(M_{mn}A_n + N_{mn}B_n) = 0.\end{aligned}\right\} \tag{33}$$

The coefficients of A_n and B_n are

$$\left.\begin{aligned} K_{mn} &= \delta_{mn}\tfrac{1}{2}L(p_n \tanh p_n + q_n \tanh q_n), \\ L_{mn} &= \frac{2\lambda^{\frac{1}{2}}(-)^{m+n}a_n}{(a_m^2/L^2+a_n^2)^2-\lambda}, \\ M_{mn} &= -L_{nm}, \\ N_{mn} &= \delta_{mn}\frac{1}{2}\left(\frac{\tanh r_n L}{r_n}+\frac{\tanh s_n L}{s_n}\right), \end{aligned}\right\} \tag{34}$$

δ_{mn} being the Kronecker delta.

Equations (33) are homogeneous and, for non-zero solutions for A_n and B_n, it is necessary that the characteristic determinant should vanish:

$$\Delta = \begin{vmatrix} 1 & P_{00} & 0 & P_{01} & \cdots \\ -P_{00} & 1 & -P_{10} & 0 & \cdots \\ 0 & P_{10} & 1 & P_{11} & \cdots \\ -P_{01} & 0 & -P_{11} & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0, \tag{35}$$

where $P_{mn} = L_{mn}/K_{mm}^{\frac{1}{2}}N_{nn}^{\frac{1}{2}}$. It is easily shown that the sum of the moduli of the non-diagonal terms form a convergent double series. It follows that the determinant (35) is convergent (Whittaker & Watson 1950, p. 37).

Now, it can be shown that, when $L \gg 1$, the characteristic determinant in (35) can be evaluated to give the first few terms of the asymptotic expansion for λ. When λ represents the lowest eigenvalue, the final result is

$$\lambda \sim \frac{3a_0^2}{L^2}\left\{1+\frac{12s}{a_0^5 L}+\left(\frac{8a_0^2}{21}+\frac{108s^2}{a_0^{10}}\right)\frac{1}{L^2}+O(L^{-3})\right\}, \tag{36}$$

where $a_0 = \frac{1}{2}\pi$ and

$$s = \frac{1}{1^5}+\frac{1}{3^5}+\ldots \approx 1\cdot 0045. \tag{37}$$

Each term in the s-series (37) arises from a term in the B_n-series in (31). Now, the form of the B_n-series is such as to give an exponentially small contribution except in a region close to each of the ends ($x = \pm L$) of the rectangular cross-section. It follows that the fluid motion near $x = \pm L$ is of boundary-layer type, the thickness of the layer being of order h. It is to be concluded that the terms involving s in (36), which increase the critical value of λ, are contributions due to these layers. As expected, the result of putting $s = 0$ in (36) gives an expression of the form (25), which is applicable when no boundary layer is present.

The analogy of the above theory with that of a thin elastic plate vibrating transversely shows that the same end-phenomenon will appear in the latter problem.

5. Experimental work

The experimental method used by Taylor (1954*b*) to measure the density gradient of a solution in a vertical tube under conditions near neutral stability, is easily adapted to the case of a vertical channel of rectangular section.

An experimental apparatus employing two tubes connected to a reservoir A is shown in figure 1. The purpose of this arrangement is to compare the density gradient at neutral stability in the rectangular channel B with the corresponding density gradient in the capillary tube C, for which the value of the critical Rayleigh number is well established. Initially, the vertical channel and the capillary tube are filled from the bottom with a transparent liquid of uniform density ρ_0. A coloured solution of slightly greater density ρ_1 is poured into the reservoir A. Since the system is dynamically unstable, convection currents are

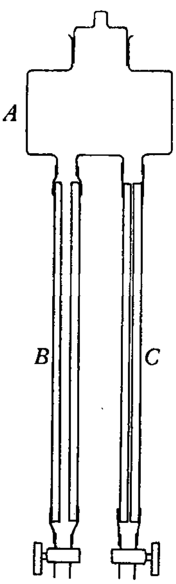


FIGURE 1

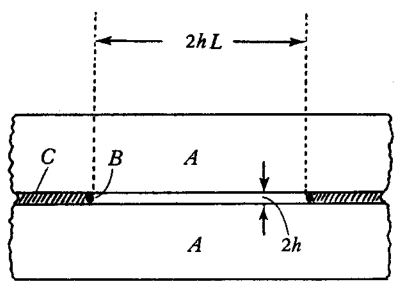


FIGURE 2

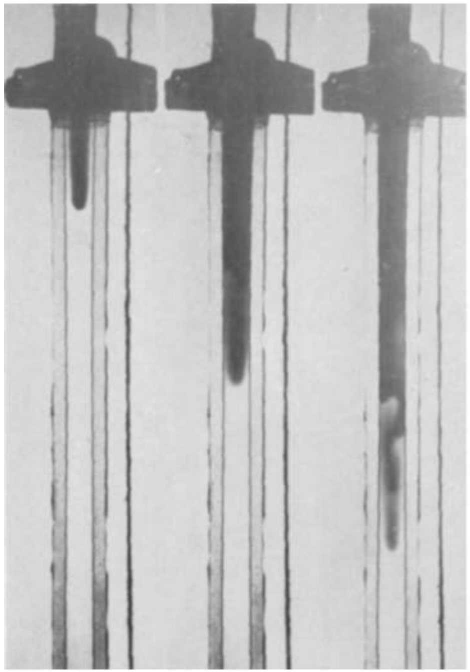
FIGURE 1. Apparatus used for the comparison of density gradients at neutral stability. A = 100 ml. reservoir, B = rectangular channel, C = capillary tube.

FIGURE 2. Cross-section through the rectangular channel. A = glass strips, B = 30 s.w.g. wire, C = cement.

set up in both vertical tubes. When equilibrium has been reached, the density of the liquid in each tube decreases with depth at very nearly the critical rate, from the value ρ_1 at the reservoir to the value ρ_0 at an appropriate point some distance down each tube. Below those points, the liquid has uniform density ρ_0. Obviously, the depth of penetration of the colouring material indicates approximately the position at which the density of the liquid has decreased to ρ_0. If the dimensions of the channel and capillary tube are known, the ratio of the respective Rayleigh numbers can be calculated from the formula

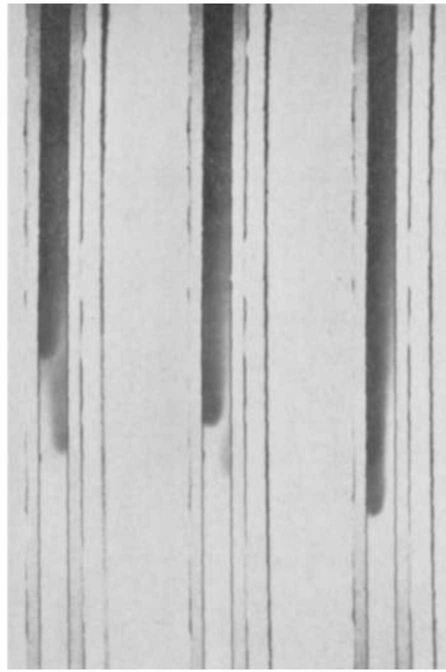
$$\frac{\lambda}{\lambda_a} = \frac{Z_a}{Z_h}\frac{h^4}{a^4}, \tag{38}$$

where λ is the critical Rayleigh number in the vertical rectangular channel, $2h$ is the spacing of the parallel planes, and Z_h is the depth of penetration of the upper liquid into the channel at neutral stability, $\lambda_a \approx 67{\cdot}94$ is the critical



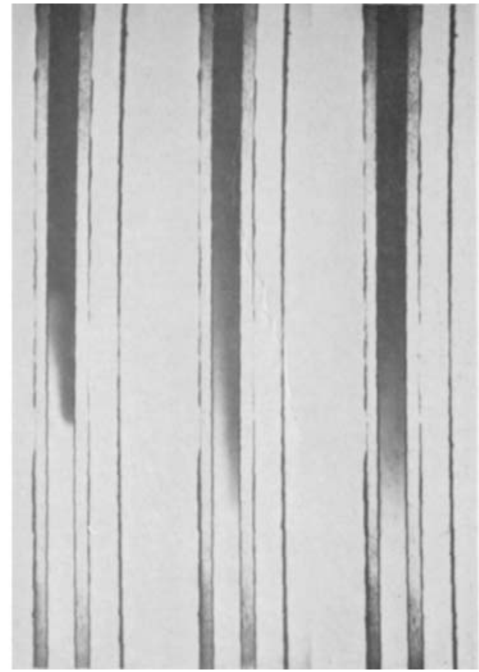
(*a*) (*b*) (*c*)

FIGURE 3 (plate 1). Three successive stages of the initial convective motion of a coloured water–glycerol mixture into a transparent mixture of slightly lower density. The times elapsed since the beginning of the experiment and the corresponding depths of maximum penetration are (*a*) 1·75 h, 3·25 cm, (*b*) 26 h, 7·8 cm, (*c*) 60 h, 11·9 cm.



(*a*) (*b*) (*c*)

FIGURE 4 (plate 1). Three intermediate stages of the convective motion of the water–glycerol mixture. The elapsed times and maximum depths of penetration are (*a*) 121 h, 16·5 cm, (*b*) 131 h, 17·2 cm, (*c*) 142 h, 18·2 cm.



(*a*) (*b*) (*c*)

FIGURE 5 (plate 1). Three stages of the final convective motion of the water–glycerol mixture. The elapsed times and maximum depths of penetration are (*a*) 181 h, 214· cm, (*b*) 227 h, 23·8 cm, (*c*) 299 h, 24·4 cm (equilibrium).

Rayleigh number in the vertical capillary tube (from theory, Taylor 1954*b*), a is the radius of the tube and Z_a is the depth of penetration of the upper liquid into the tube at neutral stability. An advantage of this comparison method is that the density, viscosity and diffusivity of the liquid need not be known when calculating the ratio λ/λ_a.

The rectangular channel used consisted of two flat glass strips about 40 cm long, which were cemented together with Araldite (a commercial two-component cement) using two stretched 30 s.w.g. wires of diameter 0·0315 cm as spacers (figure 2). This was not an ideal arrangement, as the spacing between the wires was not quite constant at all points along the channel. However, the variation in spacing was not great, and it was sufficient to take an average value in the numerical calculations. A further disadvantage arose from the fact that the ends of the internal cross-section were not truly rectangular, but, as the spacing between the two wires was more than twenty times their diameter, the effect of the shape of the ends upon the stability characteristics should be small.

The average width of the internal cross-section of the channel, shown in figure 2, was obtained by measurement of enlarged photographs; the spacing of the glass strips was obtained from micrometer measurements. In the case of the capillary tube, the internal volume was found by filling the tube with a measured volume of water, and from this the average radius was calculated. The results were $2Lh = 0{\cdot}675$ cm for the width and $2h = 0{\cdot}0320$ cm for the thickness of the cell, and $a = 0{\cdot}165$ cm for the radius of the capillary tube.

Since the critical Rayleigh number for this cell in the vertical position was of order 10^{-2}, a fairly viscous liquid was chosen—a water–glycerol mixture in equal proportions by weight, having a viscosity of about 0·06 poise. Portion of this mixture was coloured with methylene blue for addition to the reservoir at the top of the apparatus. The density ratio between the coloured and clear solutions was adjusted to about 1·002 by dissolving a small quantity of sodium chloride in a further portion of the clear mixture, and adding it drop-by-drop to each solution in turn.

During the progress of the experiment, the apparatus was kept in a darkened room in which temperature variations were small. A period of about 12 days elapsed before convective motion ceased in the rectangular channel.

After equilibrium had been reached in both the capillary tube and the rectangular channel, the respective depths of penetration of the coloured liquid were found to be $Z_a = 75{\cdot}0$ cm and $Z_h = 24{\cdot}4$ cm. When these values, together with the measured values of h and a, were substituted into (38), the result $\lambda = 1{\cdot}85 \times 10^{-2}$ was obtained.

From the given dimensions of the cross-section of the rectangular channel, the ratio $L = 21{\cdot}1$. Substituting this value into equation (36) gives $\lambda \approx 1{\cdot}77 \times 10^{-2}$, which is about 4 % lower than the experimental value. The first term of (36), which corresponds to the result derived from the simple theory of § 2, gives $\lambda \approx 1{\cdot}66 \times 10^{-2}$. This is about 10 % lower than the experimental value.

Although the contrast in the magnitude of the errors is not very great (4–10 %), it seems reasonable to explain the first of these in terms of experimental error, and to attribute the increase in the second of these to shortcomings of the simple approximate theory. In particular, the measured value of h could be in error by about 1 %, and the inaccuracy of the cell construction may have been a contributing factor. It is also possible that the measured depth of penetration of the upper liquid into the Hele-Shaw cell was underestimated due to a slow progressive loss of coloration of the methylene blue. This effect can be produced by oxidation by incident light, by reaction with other substances in solution, or by bacterial action. A further possibility, that the denser coloured fluid 'overshot' its equilibrium position in the capillary tube, has been considered, but this seems to have been unlikely in the present instance, when the density contrast ($\rho_1/\rho_0 \approx 1{\cdot}002$) was very small.

Figures 3, 4 and 5 (plate 1) illustrate the motion of the coloured liquid in the vertical rectangular channel while convection was in progress. In the early stages of the motion, a finger of coloured fluid passed down the middle of the channel, the interface near the leading edge being quite sharp (figure 3(*a*)). Before molecular diffusion had widened the coloured region appreciably, its width appeared to be equal to about one-half the width of the channel. Evidently the two liquids were behaving in a 'quasi-immiscible' manner, the finger of coloured fluid resembling those fingers observed by Saffman & Taylor (1958) in their experiments on the motion of an interface between immiscible liquids in a Hele-Shaw cell.

Figures 3(*b*) and (*c*) show the break-up of the symmetrical finger of coloured fluid. While the finger was less than about 6 cm in length, the rate of descent at the leading edge decreased approximately exponentially from an initial rate of about 3 cm/h. However, when a depth of 6 cm had been exceeded, a fairly abrupt change occurred in the character of the motion, and thereafter the rate of descent was almost constant at about 0·09 cm/h.

The change in character of the motion at a depth of 6 cm would be expected from the results of the stability theory. Since the maximum depth of penetration was about 24 cm, when the lowest mode of disturbance would be just stable, it follows that 6 cm corresponded to that depth of penetration at which the disturbance mode with twice the wave-number became stable. When the Rayleigh number, based upon the average vertical density gradient of the liquid in the cell, decreased sufficiently for the latter mode to become damped, only a finite development of the lowest mode of disturbance could be self-sustaining. The subsequent photographs (figure 4) show the new convective motion, with the heavier coloured fluid moving down one side of the channel.

The sequence of figure 4 also shows that the finite-amplitude motion of the liquid down the side of the channel was itself unstable, the predominant downward motion appearing first upon one side of the channel, and then upon the other.

In the final sequence, figure 5, the motion leading to a stable state is shown; the effect of molecular diffusion across the channel became predominant, and, in figure 5(*c*), the coloured fluid had diffused right cross the channel. That stage indicated the completion of the convective motion.

It is hoped that a more detailed treatment of the observed convective phenomena will be given in a later paper.

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